Zeta regularized products, Riemann zeta zeros and prime number spectra

G. Menezes[*]
Instituto de Física Teórica, Universidade Estadual Paulista, São Paulo, SP 01140-070, Brazil

B. F. Svaiter[†]
Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ 22460-320, Brazil

N. F. Svaiter[‡]
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, RJ 22290-180, Brazil

The Riemann hypothesis states that all nontrivial zeros of the zeta function lie in the critical line $\mathrm{Re}(s) = 1/2$. Hilbert and Pólya suggested that one possible way to prove the Riemann hypothesis is to interpret the nontrivial zeros in the light of spectral theory. Following this approach, we associate such a numerical sequence with the discrete spectrum of a linear differential operator. We discuss a necessary condition that such a sequence of numbers should obey in order to be associated with the spectrum of a linear differential operator of a system with countably infinite number of degrees of freedom. The sequence of nontrivial zeros is zeta regularizable. Then, functional integrals associated with hypothetical systems described by self-adjoint operators whose spectra is given by the sequence of the nontrivial zeros of the Riemann zeta function could be constructed. In addition, we demonstrate that if one considers the same situation with primes numbers, the associated functional integral cannot be constructed, due to the fact that the sequence of prime numbers is not zeta regularizable.

PACS numbers: 02.10.De, 11.10.-z

I. INTRODUCTION

The Riemann zeta function $\zeta(s)$ defined by analytic continuation of a Dirichlet series has a simple pole with residue 1 at $s = 1$, trivial zeros at $s = -2n$, $n = 1, 2, ...$ and infinitely many complex zeros $\rho = \beta + i\gamma$ for $\beta, \gamma \in \mathbb{R}$ and $0 < \beta < 1$. The Riemann hypothesis is the conjecture that $\beta = 1/2$ [1]. Another conjecture is that all zeros are simple. Both conjectures are unsolved problems in mathematics and much effort has been expended using various approaches to prove or disprove such conjectures. Hilbert and Pólya suggested that one way to prove the Riemann hypothesis is to give a spectral interpretation to nontrivial zeros of $\zeta(s)$. The nontrivial zeros could be the eigenvalues of a linear self-adjoint operator in an appropriate Hilbert space. For a nice introduction to the Riemann hypothesis, see the Refs. [2, 3].

In recent years the behavior of solutions of the Hamilton equations for different systems have been analyzed using methods of nonlinear mechanics. Methods in semiclassical physics allow us to formulate the quantization from the periodic orbits of a classically chaotic system. Research in this direction started with Gutzwiller [4, 5] who presented a formula which enables one to calculate the spectral density of chaotic systems, a trace formula that express the density of states as a sum over the classical periodic orbits of the system. Subsequent works of Bohigas, Giannoni, Schmit [6, 7] and also Berry [8] raise the conjecture that a quantum energy spectra of classically chaotic systems may show universal spectral correlations which are described by random matrix theory [9, 10]. Another important conjectured was stated by Montgomery [11] and supported by numerical evidences [12]: the distribution of spacing between nontrivial zeros of the $\zeta(s)$ function is statistically identical to the distribution of eigenvalue spacings in a Gaussian unitary ensemble, a result that prompted many authors to consider the Riemann hypothesis in the light of random matrix theory and quantum mechanics of classically chaotic systems [13–17]. Being more specific, there is a similarity between the number theoretical relationship that can be obtained for the nontrivial zeros of the Riemann zeta function and the sequence of prime numbers, and the connection that can be obtained between energy levels and a quantum chaotic system and classical periodic orbits. The density of Riemann zeta function zeros can be written as a trace formula where the role of periodic orbits of chaotic systems is played by the prime numbers.

Since non-separability in wave problems leads us to chaotic systems in the limit of short wavelengths, from the above discussion one cannot disregard the possibility that a quantum field model described by a nonseparable wave equation for a given boundary condition is able to reproduce the statistical properties of the nontrivial zeta zeros. Other possibility to shed some light in the spectral interpretation for the zeros is the study of field theory in disordered medium, such as wave equations in random fluids or amorphous solids [18–22] which can be modeled using random matrix theory [23, 24] or also nonlinear dielectrics [25]. For instance, it was shown that the level

[*] gsm@ift.unesp.br
[†] benar@impa.br
[‡] nfuxsvai@cbpf.br

spacing distribution of disordered fermionic systems can be described using random matrix theory [26, 27]. Therefore, further progress on the Hilbert-Pólya approach can be achieved investigating such a kind of systems discussed above. The necessary condition that a system with countably infinite number of degrees of freedom should satisfy is that its functional integral must be zeta regularized. We would like to point out that systems with the spectra of the nontrivial zeros of the $\zeta(s)$ function have been discussed in the literature before. The statistical properties of a Fermi gas whose single-particle energy levels are given by these zeros were investigated in [28, 29]. Also, in order to solve questions related to the number theory using statistical-mechanics methods, some authors introduced number theory and prime numbers in field theory [30–36].

The aim of the present paper is twofold. First we prove that the sequence of the nontrivial zeros of the Riemann zeta function can in principle be interpreted as being the spectrum of a self-adjoint operator which describes some hypothetical system with countably infinite number of degrees of freedom. We regularize the product of this sequence of numbers by defining the zeta regularized product associated with them. If these numbers can be associated with the spectrum of a linear differential operator, the zeta regularized product is the determinant of the linear operator [37–45].

Using arguments of duality we also investigate if the sequence of prime numbers could be the spectrum of some hypothetical physical system. We demonstrate that the sequence of prime numbers is not zeta regularizable [46]. In other words, one cannot conceive a physical system with countably infinite number of degrees of freedom with a pure prime number spectrum. Since any quantum mechanical system with n-degrees of freedom derives from n-component scalar field model in one dimension, our result put strong restrictions on the formulation of quantum mechanical systems with a pure prime numbers spectrum.

The organization of this paper is the following. In Section II we discuss briefly the Riemann zeta function and the Hilbert-Pólya conjecture. In Section III we prove that the nontrivial zeros of the Riemann zeta function is zeta regularizable. The consequences of this result are discussed. In section IV we show that the sequence of primes numbers is not zeta regularizable. In section V we discuss possible relations between the asymptotic behavior of a sequence and the analytic domain of the associated zeta function. Conclusions are given in Section VI. In the paper we use $k_B = c = \hbar = 1$.

II. THE RIEMANN ZETA FUNCTION AND HILBERT-PÓLYA CONJECTURE.

The prime numbers occur in a very irregular way within the sequences of all integers in local scales (local scales means on intervals comparable to their mean spacing). On the other hand, on large scales they are very regular. The best result that we have concerning its global distribution is the prime number theorem: if $\pi(x)$ is the number of primes less than or equal to x, then $x^{-1}\pi(x)\ln(x) \to 1$ as $x \to \infty$ [47–51].

Riemann showed how the distribution of the prime numbers is determined by the nontrivial complex zeros of the zeta function. A explicit formula with sums involving the prime numbers and other sums involving the zeros of the zeta function was presented [52–55]. We start discussing how the product of all primes appears in a representation of the Riemann zeta function $\zeta(s)$.

Let s be a complex variable i.e. $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$. For $\mathrm{Re}(s) > 1$ the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

converges absolutely, and uniformly for $\mathrm{Re}(s) \geq 1 + \delta$, for all $\delta > 0$. It is possible to show that

$$\zeta(s) = \prod_p \left(\frac{1}{1 - p^{-s}}\right), \tag{2}$$

for $p \in \mathcal{P}$ where $\mathcal{P}$ is the set of all prime numbers. The product giving by the Eq. (2) is called the Euler product. This is an analytic form of the fundamental theorem of arithmetic, since primes are multiplicative building block for the natural numbers. Eqs. (1) and (2) connect the additive structure in order to generate successive positive integers to this multiplicative structure. From the convergence of Eq. (2) we obtain that $\zeta(s)$ has no zeros for $\mathrm{Re}(s) > 1$.

The Riemann zeta function $\zeta(s)$ is the analytic continuation of the Dirichlet series defined by Eq. (1) to the whole complex plane. Its unique singularity is the point $s = 1$ at which it has a simple pole with residue 1. Moreover it satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{3}$$

for $s \in \mathbb{C} \setminus \{0, 1\}$. Let us define the entire function $\xi(s)$ as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{4}$$

Using the function $\xi(s)$, the functional equation given by Eq. (3) takes the form $\xi(s) = \xi(1-s)$. If ρ is a zero of $\xi(s)$, then by the functional equation so is $1-\rho$. Since $\bar{\xi}(\rho) = \xi(\bar{\rho})$ we have that $\bar{\rho}$ and $1 - \bar{\rho}$ are also zeros. The zeros are symmetrically arranged about the real axis and also about the critical line. Let us write the complex zeros of the zeta function as $\rho = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{C}$. The Riemann hypothesis is the statement that all γ are real. A weak form of the Riemann hypothesis, namely that there are no zeros of zeta function on the

line $\{s : \mathrm{Re}(s) = 1\}$, implies the prime number theorem. We have $\zeta(1 + i\tau) \neq 0$ for all $\tau \in \mathbb{R}$.

To proceed, let us define the von Mangoldt function, Λ which is defined as follows: $\Lambda(n) = \ln p$ if $n = p^\rho$ for $\rho \in \mathbb{N}$, and 0 otherwise, where p is a prime number. The significance of this function is that for $\mathrm{Re}(s) > 1$ if one differentiate logarithmically the Euler representation of zeta function for $\mathrm{Re}(s) > 1$ obtain

$$\begin{aligned} \frac{d}{ds} \ln \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} &= \\ &= \frac{d}{ds}\left(\sum_{m=1}^{\infty} \sum_{\{p\}} \frac{1}{m\, p^{ms}}\right) = \\ &= -\sum_{n\geq 1} \Lambda(n)\, n^{-s}. \end{aligned} \quad (5)$$

Next, let us define the Chebyshev function $\psi(x)$ which is the counting function for $\Lambda(n)$ by

$$\psi(x) = \sum_{n\leq x} \Lambda(n). \quad (6)$$

Using Hadamard's theory that allows one to define an analytic function by its zeros and singularities we can obtain

$$\frac{\zeta'}{\zeta}(s) = \beta - \frac{1}{s-1} + \frac{1}{2}\ln\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1) + \sum_{\rho}\left(\frac{1}{(s-\rho)} + \frac{1}{\rho}\right), \quad (7)$$

where ρ runs over the zeros of $\xi(s)$ counted according to multiplicity. The constant β is related to the Euler constant γ by the formula

$$\beta = -\frac{1}{2}\gamma - 1 - \frac{1}{2}\ln 2\pi. \quad (8)$$

Another important result is the Riemann-von Mangoldt explicit formula given by

$$\psi(x) = x - \lim_{T\to\infty} \sum_{|\mathrm{Im}\rho|\leq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1 - x^{-2}), \quad (9)$$

where the sum is taken over the nontrivial zeros of the Riemann zeta function. Possible multiple zeros are repeated according to their multiplicity. This formula is valid if x is not a power of a prime number, but can be generalized without any restriction upon x in a easy way.

The main point of Riemann's paper is that the two sequences, of prime numbers on one hand and the Riemann zeros on the other hand are connected. The erratic behavior of the primes is encoded in the nontrivial zeros of the zeta function. Although we also expect this erratic behavior for the zeros, these two sequence of numbers have totally distinct behavior with respect to being the spectrum of a linear differential operator. According to the Hilbert-Pólya conjecture, the nontrivial zeros of the Riemann zeta function are related to the eigenvalues of a self-adjoint operator. It is possible to show that the two-point correlation function of the zeros of the zeta function in the critical line is the two-point correlation function of the eingenvalues of a random Hermitian matrix taken from the Gaussian unitary ensemble, showing that there is a consistence with the conjecture of Pólya and Hilbert.

In the next section we prove that the sequence of nontrivial zeros of the Riemann zeta function is a zeta regularizable sequence and therefore can be associated with the spectrum of a linear differential operator of a system with countably infinite number of degrees of freedom.

III. THE SUPER-ZETA OR SECONDARY ZETA FUNCTION AND ITS ANALYTIC EXTENSIONS

Let (M, g) be a compact, Riemaniann C^∞ manifold, with metric $g = (g_{ij})$ on M. We assume that M is connected and $\dim M = d$. Let D be a generic elliptic operator acting on a neutral scalar field φ, both defined in M. We assume that the behavior of the fields at infinity is such that the compactification is possible. In order to obtain the correlation functions of the theory one should construct the generating functional $Z[h]$. Defining an appropriate kernel $K(m_0;\, x-y)$, the generating functional $Z[h]$ is formally defined by the following functional integral:

$$Z[h] = \int [d\varphi]\ \exp\left(-S_0 - \int d^dx\sqrt{g(x)}\, h(x)\varphi(x)\right), \quad (10)$$

where $g = \det(g_{ij})$ and the action that usually describes a scalar field is

$$S_0(\varphi) = \int d^dx\, d^dy\, \sqrt{g(x)}\sqrt{g(y)}\varphi(x)K(m_0;\, x-y)\varphi(y). \quad (11)$$

In Eq. (10), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_x d\varphi(x)$. The term m_0^2 is the (bare) mass squared of the model. Finally, $h(x)$ is a smooth function introduced to generate the Schwinger functions of the theory. One can define the functional $W[h] = \ln Z[h]$ which generates the connected Schwinger functions. In order to obtain a well-defined object, we need to regularize a determinant associated with the operator D, since $W[0] = -1/2 \ln\det D$. A similar situation arises when one calculates the one-loop effective action for non-Gaussian functional integrals.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonzero complex numbers. The zeta regularization product is defined as

$$\prod_{n\in\mathbb{N}} a_n =: \exp\left(-\frac{d}{ds}\zeta_a(s)|_{s=0}\right), \quad (12)$$

provided that the spectral zeta function $\sum_{n\in\mathbb{N}} a_n^{-s}$ has an analytic extension and is holomorphic at $s = 0$. Note that the definition of a zeta regularized product associated with a sequence of complex numbers depends on the choice of $\arg a_n$. Therefore some care must be taken

to deal with this problem. However, in this paper we assume a sequence of nonzero real numbers, so such a problem is absent.

In the following we are using the Dirichlet series inspired in the Minakshisundaram-Pleijel zeta function [56]. The standard technique of the spectral theory of elliptic operators implies the existence of a complete orthonormal set $\{f_k\}_{k=1}^{\infty}$ such that the eigenvalues obey: $0 \leq \lambda_1 \leq \lambda_2 \leq \, ... \, \leq \lambda_k \to \infty$, when $k \to \infty$ where the zero eigenvalue must be omitted (eigenvalues being counted with their multiplicities). In the basis $\{f_k\}$ the operator D is represented by an infinite diagonal matrix $D = \mathrm{diag}\,(\lambda_1, \lambda_2, ...)$. Therefore the generic operator D satisfies $Df_n(x) = \lambda_n f_n(x)$. The spectral zeta function associated with the operator D is defined as

$$\zeta_D(s) = \sum_n \frac{1}{\lambda_n^s}, \quad \mathrm{Re}(s) > s_0, \tag{13}$$

for some s_0. Formally we have

$$-\frac{d}{ds}\zeta_D(s)|_{s=0} = \ln \det D. \tag{14}$$

In order to regularize the determinant or the functional integral it is necessary to perform an analytic continuation of the spectral zeta function from some half-plane, i.e., for sufficient large positive $\mathrm{Re}(s)$ into the whole complex plane. The spectral zeta function must be analytic in a complex neighborhood of the origin, i.e., $s = 0$. This method can also be used non-Gaussian functional integrals when one calculates the one-loop effective action. For instance, we need to use scaling properties, i.e.,

$$\frac{d}{ds}\zeta_{\mu^2 D}(s)|_{s=0} = \ln\, \mu^2 \zeta_D|_{s=0} + \frac{d}{ds}\zeta_D|_{s=0}. \tag{15}$$

Let us first discuss the construction of zeta functions built over the Riemann zeros, i.e., the nontrivial zeros of the Riemann zeta function [57–63]. Assuming the Riemann hypothesis, if the Riemann nontrivial zeros are listed in pairs as usual, i.e., $\{\rho = 1/2 \pm i\gamma_k\}_{k=1,2,..}$, then the Dirichlet series that we are interested, the $G_\gamma(s,v)$ function for $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$ is defined as

$$G_\gamma(s,v) = \sum_{k=1}^{\infty} (\gamma_k^2 + v)^{-s},\ \mathrm{Re}(s) > \frac{1}{2},\ v > -\gamma_1^2. \tag{16}$$

This function can be extended to meromorphic function of $s \in \mathbb{C}$. Two interesting cases are $v = 0$ and $v = 1/4$. Let us discuss some properties of these functions, known as secondary zeta functions. The possibility of construct such Dirichlet series was discussed by Guinand [57]. Assuming the Riemann hypothesis, then the two functions $G_p(s)$ and $G_\gamma(s)$, defined respectively by

$$G_p(s) = \sum_{0<m \ln p<T} \frac{\ln p}{p^{m/2}}\,(m \ln p)^{-s} - \int_0^T e^{\frac{u}{2}} u^{-s} du \tag{17}$$

for $T \to \infty$, where p runs through the prime numbers and m through the positive integers, and

$$G_\gamma(s) = \sum_{\gamma>0} \gamma^{-s},\ \ \mathrm{Re}(s) > 1, \tag{18}$$

which is known in the literature as the super-zeta or secondary zeta function, have analytic continuation which are connected by the following functional equation

$$G_p(s) - \Gamma(1-s)\left[2^{1-s} - 2^{-s}\eta(1-s) - \left(2^{-s} - \frac{1}{2}\right)\zeta(1-s)\right] = -2\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right) G_\gamma(1-s), \tag{19}$$

where

$$\eta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s},\ \ \mathrm{Re}(s) > 1, \tag{20}$$

and by its analytic continuation elsewhere. Although the formula above was proved assuming the Riemann hypothesis, it is possible to find a generalization for it without assuming such an unproven hypothesis [59, 60]. It is possible to show that the zeta-function regularization is well defined for the family defined by the super-zeta or secondary zeta function. In the following we will present the analytic extension for $G_\gamma(s)$ assuming the Riemann hypothesis. Using the definition given by Eq. (18) we get

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} G_\gamma(s) = \int_0^\infty dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x}. \tag{21}$$

Let us split the integral that appears in Eq. (21) in the intervals $[0,1]$ and $[1,\infty)$, and define the functions

$$A(s) = \int_0^1 dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x} \tag{22}$$

and

$$B(s) = \int_1^\infty dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x}. \tag{23}$$

Note that $B(s)$ is an entire function. To proceed let us use that

$$\sum_{\gamma>0} e^{-\pi\gamma^2 x} = -\frac{1}{2\pi\sqrt{x}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\,e^{-\frac{(\ln n)^2}{4\pi x}} + e^{\frac{\pi x}{4}} - \frac{1}{2\pi}\int_0^{\infty} dt\, e^{-\pi x t^2}\Psi(t), \quad (24)$$

where the function $\Psi(t)$ is given by

$$\Psi(t) = \frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} + \frac{\zeta'(\frac{1}{2}-it)}{\zeta(\frac{1}{2}-it)}. \quad (25)$$

Substituting Eq. (24) in (22) we get that A-function can be written as

$$A(t) = A_1(t) + A_2(t) + A_3(t), \quad (26)$$

where

$$A_1(s) = -\frac{1}{2\pi}\int_0^1 dx\, x^{\frac{s}{2}-\frac{3}{2}}\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\,e^{-\frac{(\ln n)^2}{4\pi x}}\right), \quad (27)$$

$$A_2(s) = \int_0^1 dx\, x^{\frac{s}{2}-1}\, e^{\frac{\pi x}{4}} \quad (28)$$

and finally

$$A_3(s) = -\frac{1}{2\pi}\int_0^1 dx\, x^{\frac{s}{2}-1}\left(\int_0^{\infty} e^{-\pi x t^2}\Psi(t)\right). \quad (29)$$

Changing variables in the $A_1(s)$, i.e., $x \to 1/x$ we get

$$A_1(s) = -\frac{1}{2\pi}\int_1^{\infty} dx\, x^{-\frac{s}{2}-\frac{1}{2}}\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\,e^{-\frac{x(\ln n)^2}{4\pi}}\right). \quad (30)$$

It is clear that $A_1(s)$ is an entire function of s. Let us define $\Phi(s)$ as

$$\Phi(s) = A_1(s) + B(s). \quad (31)$$

Using Eqs. (23), (26), (28), (29) and (31) we can write expression (21) as

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}G_{\gamma}(s) = \Phi(s) + A_2(t) + A_3(t). \quad (32)$$

Since $\Phi(s)$ is an entire function and we have the integrals that define $A_2(t)$ and $A_3(t)$, the above formula is the analytic extension of the secondary zeta function. The function $G_{\gamma}(s)$ is a meromorphic function of s in the whole complex plane with double pole at $s=1$ and simple poles at $s=-1,-2,..,-(2n+1),\ldots$ Therefore $(s-1)^2 G_{\gamma}(s)(\Gamma(s))^{-1}$ is an entire function. From the above discussion we have that the spectral determinant associated with the zeta zeros is well defined, since it is possible to find an analytic continuation of $G_{\gamma}(s)$ to a meromorphic function in the whole complex s-plane and also analytic in a complex neighborhood of the origin, i.e., $s=0$.

A more detailed development of this interesting viewpoint can be found in [64]. The key point of the discussion is the central symmetry of the set of Riemann zeros, $\rho \to 1-\rho$, for $\rho = 1/2 \pm i\gamma_k$, and $\mathrm{Re}(\gamma_k) > 0$. Note that

$$\sum_{\rho}\frac{1}{\rho} = \sum_k\left(\frac{1}{\frac{1}{2}+i\gamma_k} + \frac{1}{\frac{1}{2}-i\gamma_k}\right) = \sum_{k=1}^{\infty}\frac{1}{(\gamma_k^2+\frac{1}{4})}. \quad (33)$$

The above identity motives one to introduce the following super-zeta functions of first, second and third kind:

$$Z_1(s,x) = \sum_{\rho}\frac{1}{(x-\rho)^s}, \;\; \mathrm{Re}(s) > 1, \quad (34)$$

$$Z_2(\sigma,v) = \sum_{k=1}^{\infty}\frac{1}{(\gamma_k^2+v)^{\sigma}}, \;\; \mathrm{Re}(\sigma) > \frac{1}{2} \quad (35)$$

and finally

$$Z_3(s,a) = \sum_{k=1}^{\infty}\frac{1}{(\gamma_k+a)^s}, \;\; \mathrm{Re}(s) > 1. \quad (36)$$

All of these super-zeta can be extended to a meromorphic function in the entire complex plane with a computable singular part. Note that in $Z_3(\sigma,a)$ we select the zeros in only one half-plane, therefore the analytic structure of this function is more singular then $Z_2(\sigma,v)$.

We conclude that there is a large class of hypothetical systems with countably infinite number of degrees of freedom, described by self-adjoint operators whose the spectra can be given by the sequence of the nontrivial zeros of the Riemann zeta function. The former sequences are zeta regularizable. In particular, one can use the super-zeta function $Z_3(s,a)$ in order to derive the density of states of a hypothetical system with random fluctuations.

IV. THE PRIME ZETA FUNCTION AND ITS ANALYTIC EXTENSIONS

Essentially, the same arguments presented above can be used to investigate the very special situation where the spectrum is given by the sequence of prime numbers. Before continuing, let us briefly discuss some results obtained by Leboeuf in [65]. The main quantity considered in such a reference was the average pair correlation of the prime numbers and also the distribution and autocorrelation of the counting function $\pi(x)$. It was shown that the average density $R_2(y)$ of prime pairs separated by a distance y and located around x for $x \to \infty$ is given by

$$R_2(y) \simeq (\ln x)^{-2}\left(1 - \frac{Si(2\pi x)}{\pi y}\right), \quad (37)$$

where $Si(x) = \int_0^x du\,\frac{sin(u)}{u}$ is the sine integral function. Note that by the prime number theorem, $(\ln x)^{-1}$ is

an approximation to the average density of distribution (number of primes per unit interval) in the neighborhood of a large number x, i.e., the asymptotic average density of prime numbers. The approximation holds when $y \ll x$. In the case where $y \gg 1$, the average density formula agrees with the average asymptotic pair correlation function discussed by Hardy and Littlewood [66]. The equation above expresses the fact that pairs of primes present anti-correlations i.e., on average the prime numbers repel each other at local distances. Next it was given a semi-classical interpretation to the oscillating part of the density of prime numbers. Since the prime asymptotically tends to behave as an uncorrelated sequence, it was suggested that the classical dynamics associated with the primes should be a nonchaotic system.

One can evoke arguments of duality to discuss the existence of a system with the spectrum given by the sequence of prime numbers. Actually, some authors [67–71] formulated the following question: is there a quantum mechanical potential related to the prime numbers? In the following we will present results that put strong restrictions on the existence of quantum field theory models and consequently quantum mechanical systems with a pure prime numbers spectrum. Let us examine this problem in more detail. To proceed, we have that a Dirichlet series is defined by

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{38}$$

Let us choose the coefficients a_n in the form

$$a_n = \begin{cases} 0, & \text{if n is 1} \\ 0, & \text{if n is not a prime number} \\ 1 & \text{if n is a prime number.} \end{cases}$$

From the Dirichlet series discussed above we have the prime zeta function $P(s)$, $s = \sigma + i\tau$, for $\sigma, \tau \in \mathbb{R}$, which is defined as

$$P(s) = \sum_{\{p\}} p^{-s}, \;\; \mathrm{Re}(s) > 1, \tag{39}$$

where the summation is performed over all prime numbers [72, 73]. We are using again the notation $p \in \mathcal{P}$ where $\mathcal{P}$ is the set of all prime numbers. The series defined by Eq. (39) converges absolutely when $\sigma > 1$.

It is clear that if $P(s)$ is the spectral zeta function of some operator, for our purposes we have to study the analytic extension of the prime zeta function. Using the Euler formula given by Eq. (2) we have

$$\ln\zeta(s) = -\sum_{\{p\}} \sum_{r=1}^{\infty} \frac{1}{r}\, p^{-rs}, \;\; \mathrm{Re}(s) > 1. \tag{40}$$

Using the definition of the prime zeta function we have

$$\ln\zeta(s) = \sum_{r=1}^{\infty} \frac{1}{r}\, P(rs), \;\; \mathrm{Re}(s) > 1. \tag{41}$$

At this point, let us define the inverse transform of the Euler product formula

$$\zeta(s) \prod_p (1 - p^{-s}) = 1. \tag{42}$$

The inverse transform of the expanded product gives

$$\prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{43}$$

where we introduced the Möbius function $\mu(n)$ defined by [75, 76]

$$\mu(n) = \begin{cases} 1, & \text{if n is a square-free positive integer} \\ & \text{with an even number of prime factors} \\ -1, & \text{if n is a square-free positive integer} \\ & \text{with an odd number of prime factors} \\ 0 & \text{if n is not square-free} \end{cases}$$

and $\mu(1) = 1$. Using the Möbius function $\mu(n)$ it is possible to show that the prime zeta function $P(s)$ can be expressed as

$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln\zeta(ks), \;\; \mathrm{Re}(s) > 1. \tag{44}$$

The analytic continuation of the prime zeta function can be obtained simply by using the analytic continuation of the Riemann zeta function. Defining the function $\phi(x)$ as

$$\phi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}, \tag{45}$$

we can write

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} dx\, \phi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1}{2}(s+1)} \right). \tag{46}$$

The integral that appears above is convergent for all values of s and therefore Eq. (46) gives the analytic continuation of the Riemmann zeta function to the whole complex s-plane. The only singularity is the pole at $s = 1$, since the pole at $s = 0$ is canceled by the pole of the Gamma function $\Gamma(s/2)$.

As we discussed, the Riemann zeta function $\zeta(s)$ has a pole at $s = 1$. This relationship shows that $s = 1/k$ is a singular point for all square free positive integers k. This sequence limits to $s = 0$. All points on the line $\mathrm{Re}(s) = 0$ are limit points of the poles of $P(s)$, so that the line $\mathrm{Re}(s) = 0$ is a natural boundary of $P(s)$. The prime zeta function can be analytically extended only in the strip $0 < \sigma \leq 1$. This result, that the prime zeta function cannot be continued beyond the line $\sigma = 0$ was obtained by Landau and Walfisz [72] and discussed by Fröberg [73]

and Illies [74]. Let us use that $\zeta(ks) = z$, where z is a complex variable. The function $\ln z$ is analytic at every point of its Riemann surface and satisfies $d\ln z/dz = z^{-1}$. We have that

$$\frac{d}{ds}P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \frac{1}{\zeta(ks)} \frac{\partial}{\partial s}\zeta(ks), \;\; \mathrm{Re}(s) > 0. \quad (47)$$

If k is a square-free positive integer, in the zeros of the Riemann zeta function ($\zeta(ks) = 0$), $dP(s)/ds$ diverges. If k is not a square-free integer positive integer, again the quantity $dP(s)/ds$ is not defined. Since we need to compute the derivative of the spectral prime zeta function at $s = 0$, the assumption of a prime number spectrum implies a ill-defined functional integral. Hence the sequence of primes numbers is not zeta regularizable.

Finally, more can be said. Suppose that the generic operator D is the usual Laplacian defined in a Euclidean manifold. If one compares the Weyl law for the asymptotic series for the density of eigenvalues of the Laplacian operator in a three-space, $N(\omega) = V\omega^2/2\pi^2 + ..$, where V is the volume of the three-space [77, 78] with the asymptotic distribution of the zeta zeros or the asymptotic distribution of prime numbers that follows from the prime number theorem, we get quite different regimes. For usual quantized systems we get that ω_n must increase at a rate dictated by the Weyl law. For the prime numbers, using the prime number theorem, we get the asymptotic regimes $p_n \sim n \ln n$. If the zeros $\rho = \beta + i\gamma$, with $\gamma > 0$ are arranged in a sequence $\rho_k = \beta_k + i\gamma_k$ so that $\gamma_{k+1} > \gamma_k$, then $|\rho_k| \sim \gamma_k \sim 2\pi n/\ln n$ as $n \to \infty$. Therefore for the zeros of the zeta function we get $\gamma_n \sim n/\ln n$. Although both asymptotic regimes are quite different from the usual systems described by the Laplacian, the zeta regularization works for the sequence of zeta zeros and fails for the sequence of prime numbers. In the next section we discuss possible relations between the asymptotic behavior of a numerical sequence and the analytic domain of the associated zeta function.

V. THE APPROXIMATIONS FOR THE SUPER-ZETA FUNCTION AND THE PRIME ZETA FUNCTION USING ASYMPTOTIC DISTRIBUTIONS

The aim of this section is to propose an approximation for a zeta function of a numerical sequence by means of its asymptotic distribution. For the case of the nontrivial zeros of Riemann zeta function, the analytic domain and polar structure of the approximated zeta function is the same as the super-zeta function for $\mathrm{Re}(s) > -\frac{3}{2}$. However, for the sequence of prime numbers, this approximation fails to reproduce the analytic structure of the prime zeta function.

Let $\lambda_1 < \lambda_2 < \ldots$ be the spectrum of a linear operator D. We discussed before a necessary condition for the existence of a physical system with countably infinite degrees of freedom and possessing such a spectrum: the spectral zeta-function associated with such a sequence shall have an analytic continuation which includes the origin. What properties of the sequence (λ_n) determine the existence of this analytic continuation?

The spectral zeta function associated with the sequence λ_n can be expressed as

$$\zeta_D(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} = \lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n^{-s}. \quad (48)$$

In order to define the above zeta function by means of an integral, let us introduce the following counting function

$$F(t) = \#\{\lambda_n \,|\, \lambda_n < t\}$$

that is, $F(t)$ is the number of elements in the sequence λ_n which are less than t. The direct use of the definition of the Riemann-Stieltjes integral yields

$$\sum_{n=1}^{m} \lambda_n^{-s} = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_a^b t^{-s}\, dF(t),$$
$$\lambda_{k-1} \le a < \lambda_k \,, \lambda_m \le b < \lambda_{m+1}. \quad (49)$$

Therefore, the spectral zeta function $\zeta_D(s)$ can be expressed as

$$\zeta_D(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_{\lambda_k}^{\infty} t^{-s}\, dF(t). \quad (50)$$

Such a formula is valid in the region of convergence of the series giving by Eq. (48). Since the finite sum in the right-hand side of Eq. (50) is analytic over the whole complex s-plane, the qualitative behavior of the analytic extension of the above function is determined by the above Riemann-Stieltjes integral.

Since it is a hard task to compute $F(t)$ in general, one may resort to asymptotic expansions, when they are available. For the case where the sequence λ_n is given by the sequence of nontrivial zeros of the Riemann zeta function, we have

$$F(t) \approx t\ln t \,, t \to \infty. \quad (51)$$

Using this approximation to $F(t)$ we get, for the spectral zeta function of the nontrivial Riemann zeros

$$\tilde{\zeta}_D(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_{\lambda_k}^{\infty} t^{-s}\, d(t\ln t). \quad (52)$$

Since the integrator $t\ln t$ is smooth over the interval $(0, \infty)$, the Riemann-Stieltjes integral coincides with the usual Riemann integral and we have

$$\int_{\lambda_k}^{\infty} t^{-s}\, d(t\ln t) = \int_{\lambda_k}^{\infty} t^{-s}\, \frac{d}{dt}(t\ln t)\, dt.$$

Thus, one gets for $\mathrm{Re}(s) > 1$

$$\tilde{\zeta}_D(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + \frac{\lambda_k^{-(s-1)}}{(s-1)^2} + \frac{\lambda_k^{-(s-1)}}{(s-1)}\,(1 + \ln\lambda_k)\,. \quad (53)$$

Analytic continuation of the above result gives the same pole structure of the super-zeta $G_\gamma(s)$ in the neighborhood of $\mathrm{Re}(s) \geq 0$.

Now let us apply the same reasoning when the sequence λ_n is given by the sequence of primes. In this case, the integrator $F(t)$ is just the prime counting function $\pi(t)$. Asymptotically we have

$$F(t) = \pi(t) \approx \frac{t}{\ln t}\,, t \to \infty. \tag{54}$$

Using this approximation, the spectral zeta function for the sequence of prime numbers is given by

$$\hat{\zeta}_D(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + \int_{\lambda_k}^{\infty} t^{-s}\, d\left(\frac{t}{\ln t}\right). \tag{55}$$

The integrator is smooth over the interval (λ_k, ∞), $\lambda_k > 1$, then the again Riemann-Stieltjes integral coincides with the usual Riemann integral. The final result is

$$\begin{aligned}\hat{\zeta}_D(s) = & \sum_{n=1}^{k-1} \lambda_n^{-s} + \mathrm{E}_1[(s-1)\ln\lambda_k] \\ & - \frac{1}{\ln\lambda_k}\mathrm{E}_2[(s-1)\ln\lambda_k],\end{aligned} \tag{56}$$

where $\mathrm{E}_m(z)$ is the usual exponential integral. Using the recurrence relation

$$E_{n+1}(z) = \frac{1}{n}[e^{-z} - z\,E_n(z)] \;\; n = 1, 2, 3.. \tag{57}$$

we have

$$\hat{\zeta}_D(s) = \sum_{n=1}^{k-1} \lambda_n^{-s} + s\,E_1[(s-1)\ln\lambda_k] - \frac{1}{\lambda_k^{s-1}\,\ln\,\lambda_k}. \tag{58}$$

Using the analytic continuation of such a function in the above result yields multi-valued functions with a branch point at $s = 1$. Comparison with the prime zeta function $P(s)$ defined by Eq. (39) reveals that $\hat{\zeta}_D(s)$ fails to reproduce qualitatively the same pole structure of $P(s)$ in the neighborhood of $\mathrm{Re}(s) \geq 0$.

VI. CONCLUSIONS

The analytic function which encodes information on the prime factorization of integers and distribution of primes is the Riemann zeta function $\zeta(s)$ because in the region of the complex plane where it converges absolutely and uniformly, the product of all prime numbers appears in a representation of the Riemann zeta function $\zeta(s)$. The Riemann hypothesis claims that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. This hypothesis made a deep connection between primes numbers and zeros of analytic functions. Hilbert and Pólya suggested that there might be a spectral interpretation of the the non-trivial zeros of the Riemann zeta function. The corresponding operator must be self-adjoint.

There are many evidences that support this point of view. First, there is an explicitly formula derived by Weil that connects the Riemann zeros with prime numbers. This formula is similar to the trace formula derived by Gutzwiller, which relates quantum energylevels to classical periodic orbits in chaotic systems. Also there is the conjecture that a quantum energy spectra of classically chaotic systems may show universal spectral correlations which are described by random matrix theory. Finally the crucial result obtained by Montgomery for the pair correlation of the nontrivial zeros connects these subjects.

Non-separability in wave problems leads us to chaotic systems in the limit of short wavelengths. Therefore one cannot disregard the possibility that a quantum field model described by a nonseparable wave equation for a given boundary condition is able to reproduce the statistical properties of the nontrivial zeta zeros. Other possibility is to consider a field theory in disordered medium, such as wave equations in amorphous solids We conjecture that a system with countably infinite number of degrees of freedom with randomness can be used to achieve further progress in the Hilbert-Pólya conjecture.

In this paper we obtained that in principle hypothetical physical systems described by self-adjoint operators with a spectrum given by the sequence of nontrivial zeros of the Riemann zeta function can be realized in nature. We also proved that for systems described by a self-adjoint operator where the spectrum is given by the prime numbers, the associated functional integral cannot be constructed. The impossibility of extending the definition of the analytic function $P(s)$ to the half-plane $\sigma < 0$ means that the functional determinant cannot be defined, in virtue that the prime numbers sequence is not zeta regularizable. Accordingly, the results obtained here put series restrictions on the formulation of quantum mechanical systems with a pure prime numbers spectrum.

If one compares the Weyl law for the asymptotic series of the density of eigenvalues of the Laplacian operator in a three-space with the asymptotic distribution of the zeta zeros or the asymptotic distribution of prime numbers, we get quite different regimes. For the prime numbers, using the prime number theorem, we get the asymptotic regimes $p_n \sim n \ln n$. For the zeros for the zeta function we get $\gamma_n \sim n/\ln n$. Although both asymptotic regimes are quite different from the usual systems, it is certainly a step forward to explain why the zeta regularization works for the sequence of zeta zeros and fails for the sequence of prime numbers.

VII. ACKNOWLEGEMENTS

This paper was supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnológico do Brazil (CNPq).

[1] B. Riemann, *"Über die Anzahlde Primzahler unter einer gegebenen Grösse"*, Monatsberichte der Berliner Akademie, 671 (1859).
[2] *The Riemann Hypothesis*, edited by P. Borwein, S. Choi, B. Rooney and A. Weirathmueller (Springer Science+Bussines Media LLC, New York, 2008).
[3] D. Schumayer and D. A. W. Hutchinson, Rev. Mod. Phys. **83**, 307 (2011).
[4] M. G. Gutzwiller, J. Math. Phys. **11** 1791 (1970).
[5] M. G. Gutzwiller, J. Math. Phys. **12** 343 (1971).
[6] O. Bohigas, M. J. Giannoni and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[7] O. Bohigas and M. J. Gianonni, in *Mathematical and Computational Methods in Nuclear Physics*, edited H. Araki, J. Ehlers, K. Hepp, R. Kippenhahn, H. Weidenmuller and J. Zittart (Springer Verlag, Berlin, 1984).
[8] M. V. Berry, in *Chaotic Behavior of Deterministic Systems*, edited by G. Iooss, R. Helleman and R. Stora (North Holland, Amsterdam, 1981), Le Houches Section XXXVI.
[9] F. J. Dyson, Jour. Math. Phys. **3**, 140 (1962).
[10] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 2004).
[11] H. L. Montgomery, Proc. Symp. Pure Math. **24**, 181 (1973).
[12] A. M. Odlyzko, Math. Comp. **48**, 273 (1987).
[13] E. B. Bogolmony and J. P. Keating, Nonlinearity, **8**, 1115 (1995).
[14] E. B. Bogolmony and J. P. Keating, Nonlinearity, **9**, 911 (1996).
[15] M. V. Berry and J. P. Keating, Siam Rev. **41**, 236 (1999).
[16] J. P. Keating and N. C. Snaith, Comm. Math. Phys. **214**, 57 (2000).
[17] P. Bourgade and J. P. Keating, Séminaire Poincaré **XIV**, 115 (2010).
[18] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
[19] *Scattering and Localization of Classical Waves in Random Media*, edited by P. Sheng (World Scientific, Singapore, 1990).
[20] J. D. Maynard, Rev. Mod. Phys. **73**, 401 (2001).
[21] G. Krein, G. Menezes and N. F. Svaiter, Phys. Rev. Lett. **105**, 131301 (2010).
[22] E. Arias, E. Goulart, G. Krein, G. Menezes and N. F. Svaiter, Phys. Rev. D **83**, 125022 (2011).
[23] P. J. Forrester, N. C. Snaith and J. J. M. Verbaarschot, J. Phys. A **36**, R1 (2003).
[24] *Application of Random Matrices in Physics*, edited by E. Brézin, V. Kazakov, D. Serban, P. Wiegmann and A. Zabrodin (Springer, Netherlands, 2006).
[25] L. H. Ford, V. A. De Lorenci, G. Menezes and N. F. Svaiter, Ann. of Phys. **329**, 80 (2013).
[26] K. B. Efetov, Sov. Phys. JETP **55**, 514 (1982).
[27] K. B. Efetov, Sov. Phys. JETP **83**, 833 (1982).
[28] P. Leboeuf, A. G. Monastra and O. Bohigas, Regular and Chaotic Dynamics **2**, 205 (2001).
[29] P. Leboeuf, A. G. Monastra, Nucl. Phys. A **724**, 69 (2003).
[30] B. Julia, in *Number Theory and Physics*, edited by J.-M. Luck, D. Moussa and M. Waldschmidt (Springer Verlag, Berlin, 1990).
[31] D. Spector, Comm. Math. Phys. **127**, 239 (1990).
[32] I. Bakas and M. J. Bowick, J. Math. Phys. **32**, 1881 (1991).
[33] M. J. Bowick, in *Thermal Field Theory* edited by H. Ezawa, T. Arimitzu and Y. Hashimoto (Elsevier Science Publishers, B.V., 1991).
[34] B. L. Julia, Physica A **203**, 425 (1994).
[35] D. Spector, Comm. Math. Phys. **177**, 13 (1996).
[36] D. Spector, J. Math. Phys. **39**, 1919 (1998).
[37] R. T. Seeley, Amer. Math. Soc. Proc. Symp. Pure Math. **10**, 288 (1967).
[38] D. B. Ray and I. M. Singer, Advances in Math. **7**, 145 (1971).
[39] J. S. Dowker and R. Crithley, Phys. Rev. D **13**, 3224 (1976).
[40] S. W. Hawking, Comm. Math. Phys. **55**, 133 (1977).
[41] A. Voros, Comm. Math. Phys. **110**, 439 (1987).
[42] S. K. Blau, M. Visser and A. Wipf, Nucl. Phys. B **310**, 163 (1988).
[43] A. Voros, Adv. Stud. Pure Math. **21**, 327 (1992).
[44] J. R. Quine, S. H. Heydary and R. Y. Song, Trans. Amer. Math. Soc. **338**, 213 (1993).
[45] G. V. Dune, J. Phys. A **41**, 1 (2008).
[46] G. Menezes and N. F. Svaiter, *"Quantum Field Theories and Prime Numbers Spectrum"* [arXiv:1211.5198 [math-ph]].
[47] J. Hadamard, Bull. Soc. Math. France **24**, 119 (1986).
[48] C. de Valle-Poussin, Ann. Soc. Sci. Bruxelles **202**, 183 (1896).
[49] C. de Valle-Poussin, Ann. Soc. Sci. Bruxelles **202**, 281 (1896).
[50] D. J. Newman, Am. Math. Month. **87**, 693 (1980).
[51] J. Korevaar, Math. Intell. **4**, 108 115 (1982).
[52] A. E. Ingham, *The Distribution of Prime Numbers* (Cambridge University Press, Cambridge, 1990).
[53] E. C. Titchmarsh, *The Zeta-function of Riemann* (Stechert-Hafner Service Agency, New York, 1964)
[54] S. J. Patterson, *An introduction to the theory of the Riemann Zeta-Function* (Cambridge University Press, Cambridge, 1988).
[55] G. Tenenbaum and M. M. France, *"The Prime Numbers and Their Distribution"*, American Mathematical Society (2000).
[56] S. Minakshisundaram and M. Pleijel, Can. J. Math. **1**, 242 (1949).
[57] A. P. Guinand, Proc. Lond. Math. Soc., **50**, 107 (1945).
[58] J. Delsarte, Jour. Anal. Math. **17**, 419 (1966).
[59] I. C. Chakravarty, Jour. Math. Anal. Applic. **30**, 280 (1970).
[60] I. C. Chakravarty, Jour. Math. Anal. Applic. **35**, 484 (1971).
[61] A. Ivic, Bull. CXXI l'Acad. Scien. Arts, **26**, 39 (2003).
[62] A. Voros, Ann. Inst. Fourier, Grenoble **53**, 665 (2003); erratum **54** 1139 (2004).
[63] A. Voros, in *Frontiers in Number Theory, Physics and Geometry*, edited by P. Cartier, B. Julia, P. Moussa and P. Vanhove (Springer Verlag, Berlin, 2006).
[64] A. Voros, *Zeta Functions over Zeros of Zeta Functions* (Springer-Verlag, Heidelberg, 2010).
[65] P. Leboeuf, Ann. Henri Poincaré **4**, S727 (2003).

[66] G. H. Hardy and J. E. Littlewood, Acta Math. **44**, 1 (1923).
[67] G. Mussardo, *"The Quantum Mechanical Potential for the Prime Numbers"* [9712010[cond-math]].
[68] H. Rosu, Mod. Phys. Lett. **18**, 1205 (2003).
[69] J. Sakhr, R. K. Bhaduri and B. P. van Zyl, Phys. Rev. E **68**, 026206 (2003).
[70] S. Joffily, Nucl. Phys. A **790**, 631c (2007).
[71] D. Schumayer, B. P. van Zyl, D. A. W. Hutchinson, Phys. Rev. E **78**, 056215 (2008).
[72] Von E. Landau and A. Walfisz, *"Über die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen"*, Rendiconti del Circolo Matematico di Palermo **44**, 82 (1919).
[73] C. E. Fröberg, BIT, **8**, 187 (1968).
[74] G. Illies, Comm. Math. Phys. **220**, 69 (2001).
[75] K. Knopp, *Theory and Applications of Infinite Series* (Dover Publications, Inc., New York, 1990).
[76] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford Clarenton Press, London, 1956).
[77] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953).
[78] D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979).